THE
MATHEMATICAL GAZETTE.

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OUR LOCAL BRANCH.

The summer meeting of the North Wales branch was held on 22nd May at the County School, Beaumaris, where the headmaster, Mr. Madoc Jones, entertained the members present. In opening a discussion on the teaching of geometry, Mr. Botting, mathematical tutor at the Normal College, Bangor, referred to the recent circular of the Board of Education, which recommends three stages, each occupying one year: (1) children to be grounded in fundamental ideas of space, lines, angles, etc., without actual definitions, and accustomed to instruments of measurement; (2) the acquisition of a number of fundamental propositions, but not in a formal way, *e.g.* I. 13-15 in Euclid, and some propositions about parallel lines, the equality of triangles, etc.; (3) rigid deductive proofs to be begun, all fresh propositions being treated as original work like riders. He also said that propositions should not be taken singly, but in groups together, and that the power of working riders is the only real test of advance in geometry.

In the subsequent discussion nearly all the speakers agreed with Mr. Botting about the difficulty of getting the average pupil to work riders, even the simpler ones set in examination papers, and considered that many of those set were too hard. The chairman, Prof. Bryan, and others, thought it best to start with mensuration of areas, volumes, etc., rather than with pure knowledge, such as the angles of a triangle being equal to two right angles. Several expressed their preference for the old teaching from Euclid's text as being more logical than the methods now adopted.

Mr. Ferguson, science lecturer at Bangor University College, after urging the claims of mathematical analysis over those of geometry, read part of an interesting paper on the progress of analysis since ancient times. He referred to the early Diophantine problems, the special methods formerly used for solving special problems, and the lack of symbolism; and proceeded to the solution of cubic equations in the 16th century by Floridus, Tartaglia and Cardan. This subject will be continued at the next meeting in November.

A PROPOSAL FOR THE UNKNOWN DIGIT.

In last year's address I called attention to the need of some symbol to denote unknown digits in approximate numerical calculations. The present note is communicated in the hopes

that the matter may be submitted either to the Council or to a special committee of the Association in order that some definite recommendation may be adopted.

When we speak of the sun's distance from the earth as 92,000,000 miles the last six 0's do not stand for zeros. We do not even imply that the distance is necessarily not less than 91,999,999 miles or greater than 92,000,001 miles. What the statement really implies is that the first two significant figures are 92 and that these stand respectively in the tens of millions and the millions places. The use of zeros to denote the remaining digits is thus misleading. A further ambiguity arises when the last known digit of a number is actually a zero. Thus taking the earth's radius as 3963 miles, the result is

4000	miles to one	significant	figure,	
4000	„	two	„	figures,
3960	„	three	„	„
3963	„	four	„	„

but when we only write 4000, there is nothing to show whether the result is true to one, two or three significant figures. There is nothing to distinguish the 0 which comes after the 4 from the other two 0's, although one should represent a zero and the other two undetermined digits.

It is of course possible to suggest any number of different symbols for the unknown digit. Two which seem not altogether unsuitable are the note of interrogation ? and a zero with a bar through the middle θ. The latter is not altogether unlike a Greek θ, and might be written in the same way as that letter. The following samples show the two notations:

Sun's distance		92,θθθ,θθθ	92,???,???
Earth's rad. to	2 figs.	4,θθθ	4,0??
„ „	3 figs.	3,96θ	3,96?
„ „	4 figs.	3,963·θ	3,963·?

The objection which a colleague made on hearing these suggestions took the form of the question: If you speak of 7,2θθ do you mean a number between 7,150 and 7,249, or a number between 7,200 and 7,299, or even between 7,101 and 7,200, inclusive in every case. My friend seemed to think that *three* new symbols would be required. Surely that was no reason for objecting to one new symbol. His contention would imply that at present 0 stands for four different things. As a matter of fact the remaining ambiguities can be got over by appending the signs $\pm$, or $+$ or $-$ after the numbers whenever there is any doubt as to which meaning is to be assigned to the approximation.

The natural name for the unknown digit is, as I suggested, "aught" in contradistinction to "nought." If preferred the

word "any" might be used instead, thus 236θ would read two three six any.

If the 0 with the horizontal bar is decided on, it may be useful at times to introduce other figures similarly ruled through. For example, in multiplying 3θ by 6, if we say six times aught are aught and six times three are eighteen, writing the result 18θ, we have omitted to allow for carrying on from the multiplication of θ by 6. On the other hand if we say six times aught are aught carry aught, six times three are eighteen and aught are aught carry two and write the answer 2θθ, we are safe at any rate, but we are not making such a close estimate as if we wrote the answer 18θ.

The resemblance of θ to the Greek θ is rather useful in connection with the custom of using θ to denote an unknown quantity between 0 and 1 in the remainder in Taylor's Series. With the new notation $f^{(n)}(x+\theta h)$ would become $f^{(n)}(x+(0{\cdot}\mathrm{\theta}+)h)$.

It is impossible to add, subtract, multiply or divide past the unknown digit. The latter acts as an automatic brake which stops the pupil from evaluating a long string of digits with his customary unmathematical inaccuracy.

The notation $1{\cdot}57\times10^6$ certainly gets over the difficulty to a limited extent, but it is difficult to make people use this notation.

There is also the need of a notation for an indeterminate quantity; unfortunately the symbol most suited for the purpose is used to denote "per cent."

The suggestions raised here will doubtless give rise to differences of opinion. Possibly a better notation than either I have suggested can be found. For this reason I would urge the Association to deal with the matter, either by appointing a small committee or by referring the question to the Council.

G. H. BRYAN.

AN ARBITRARY VETO.

HAVING read with more than ordinary interest the Address delivered by Professor Perry on "The Correlation of the Teaching of Mathematics and Science," I should like to lay before the members of the Mathematical Association a few remarks which, I trust, will be found to the point.

I need not further endorse the view that as long as a student does not specialize in mathematics or in science, the teaching of both these subjects will be greatly improved if in the hands of the same teacher. Such an arrangement must benefit both teacher and student. By "science" I mean physics, and those parts of applied mechanics, electricity, and magnetism, generally comprised with certain other practical subjects under the general term of "engineering."

It is inevitable that those who have to teach the working-class element that forms the majority of those who attend Evening Classes should hold this view.

To these students,—wage-earning lads, and thereby somewhat inclined to be more independent and critical, and less amenable to school routine than the ordinary school-boy—the logic of the pure mathematician will scarcely appeal. And further, the practical interest a physicist or an engineer can give to most of the questions that are considered in mathematics is certain to react upon them and to promote that attention and regularity in attendance without which no teaching can have really good and lasting results.

Take the case of $(1+a)^n=1+na$, when a is small compared to unity. The mathematician will see but an occasion for abstract applications and will accordingly keep his students on such exercises as evaluating $(1{\cdot}006)^3$ or $\sqrt[3]{0{\cdot}994}$. But the mathematical physicist or engineer will seize the opportunity of introducing practical examples. He will, for instance—even if this necessitates the definition of "pitch" and of "arc of approach," which he can easily explain in a few words (and by means of two sectors of gearing cut in cardboard, if he be not taken unawares)—mention that, with the usual proportion of involute gearing, the ratio of the arc of approach to the pitch is found to be

$$\frac{n}{24}\left(\sqrt{1+\frac{57{\cdot}6}{n}}-1\right)$$

where n is the number of teeth in the follower. He will then ask his students to show that, when the number of teeth is very much greater than 57·6, the ratio is independent of the former. Such examples will teach the students not only what the tool is, but for what purpose it is used, and how it is used, a point which is too often entirely lost sight of in teaching mathematics.

The mathematician will set his class to plot such functions as, say, $y=\sqrt{\cdot 5-\dfrac{1}{10\sqrt{x}}}$. The scientist will not lose such a chance of interesting his audience. He will give a short exposition of the principle discovered by Torricelli, so beautiful in its simplicity, and leading to the expression for the discharge of water through a drowned orifice:—$Q=cw\sqrt{2gh}$; (which lends itself admirably to the purpose of physical interpretation). He will then explain in what consists the Spanish module and will then easily deduce for the annular orifice of the module the formula $r=\sqrt{R^2-\dfrac{Q}{a\sqrt{h}}}$, where R, Q, a will be numerical constants, while r and h are the correlated variables. Each student will then set to work and plot the outline of the plug,

and will have learned ten times more than a bare graph in x, y would have conveyed to him. In addition, this (so precious) conviction that mathematics are not a thing "which they will have to do only in hell" will be profoundly strengthened. (If the opportunity allows, this should at once be followed by the graphical determination of the volume and therefore of the weight of the plug.) To exercise their knowledge as to the graphic solution of equations, instead of keeping students on such work as: "Solve $x^3-5x^2+2=0$," a physicist will tell them: "I want to make an anchor ring, the weight must be 9·68 lbs., the outer diameter being 10 inches; the metal weighs ·245 lbs. per cubic inch; how thick must it be?" Mathematics will veil her face, but what a gain for the boys! They will first find out that they are led to a cubic, and be puzzled; one single word, "graph," will launch them excitedly on the pursuit of the required radius of cross-section by plotting $y=x^3-5x^2+2$. Some will find for answer 4·92 in., others 0·68 in. Thus will they learn the never-to-be-forgotten lesson that one must put a pennyworth of salt with results, obtained by mathematical methods, which are to be put to a practical use.

I could multiply examples, but the cases selected show the greatly increased power of a teacher to impart instruction of the right sort when he can call to the rescue work done or touched upon in other (science) classes taken by himself.

I was greatly surprised that while most speakers approved of the correlated teaching of mathematics and science by the same teacher, none alluded to the arbitrary veto of the University of London. For a teacher to be recognized by the University (and many posts are granted but on this condition) the teacher must not deal with any subject other than the one in which he wishes to be recognized. Before anything can be done this regulation must be abolished. It is responsible for a great deal of the rigour that is keeping strictly apart the various departments of a College or a Polytechnic.

My own experience is bad enough, but as it is a very good example of the case under discussion I may be excused for stating it at length.

One of my duties is to teach apprentices who, after passing a preliminary easy examination, enter a large factory. They come from various schools, have been taught in various ways (some very badly indeed), and to all, without exception, the word "accuracy" is Hebrew. This heterogeneous material is given to me to turn in one year into as homogeneous a body of students as possible, so as to enable a more advanced course to be successfully followed in the second year. As it may be inferred, the task is highly interesting.

Four years ago, I took them in pure mathematics, practical

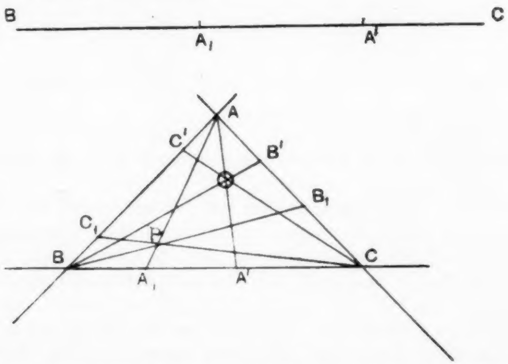
mathematics (lectures and exercise classes) and general elementary science (lectures and laboratory work). Having these three subjects in hand, I found the work fascinating, the classes being so related that each was a part of a whole, one giving the practical illustrations required in the apparently quite independent theoretical work of another. The results, as may be expected under such a system, were most gratifying, the more so that, being practically their only teacher, I got to be "in touch" with the boys, and this means much when, because they earn a small salary, they consider themselves as beyond the school-boy stage. It was very pleasant work indeed, and I was taking steps to obtain a free pass to visit weekly the shops in which the boys were employed, in order to bring into the evening work something directly connected with their own daily experiences, and thus to make the general educational scheme as efficient as could be. But, owing to difficulties presented by the above-mentioned University regulation, the classes were split up, and I was left with the teaching of mathematics alone.

My experience cannot be unique. If such promising conditions are to be sacrificed on the altar of the god Classification, no improvement in this line can be hoped for as long as the University of London adheres to the present rule concerning recognized teachers.

MAURICE E. J. GHEURY.

ANHARMONIC COORDINATES.

HAMILTON, in his *Elements of Quaternions* (Book I. Chapter II.), developed this system of coordinates, by means of his vectorial algebra, in connection with his theory of geometrical nets.



In the following paper I have re-proved the fundamental theorems, by other methods, and have sketched out the natural lines of development of the system.

Though few, if any, of the results arrived at are new, they are at any rate newly stated; and I believe that the elegance of method will command interest.

In another paper, on Generalised Coordinates, I shall show that all systems of coordinates can be immediately reduced to Anharmonic Coordinates.

I. Definition $$(BA'CA_1) \equiv \frac{BA' \cdot CA_1}{A'C \cdot A_1B}.$$

II. (1) Anharmonic Coordinates. A, B, C, O fixed points.

Now $$A(BOCP) = (BA'CA_1).$$

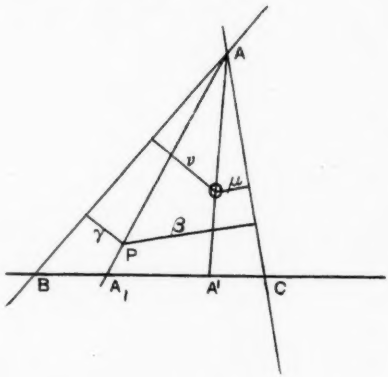
Hence by multiplying together Ceva's equations for O and P, we obtain

$$A(BOCP) \cdot B(COAP) \cdot C(AOBP) = 1.$$

Hence P is uniquely determined by any two of these ranges, and they may be respectively symbolised by $\frac{y}{z}, \frac{z}{x}, \frac{x}{y}$.

And (x, y, z) may be called the coordinates of P for the system of points A, B, C, O.

(2) $\left.\begin{array}{l}\text{The coordinates of } A \text{ are } (1, 0, 0). \\ \text{And of } O \text{ are } (1, 1, 1). \\ \text{And of } A' \text{ are } (0, 1, 1).\end{array}\right\}$



III. (1) Reduction to Trilinears.

Let the trilinear coordinates of O and P be (λ, μ, ν), (α, β, γ) respectively.

Then $\frac{y}{z} = A(BOCP) = \frac{\sin BAO \cdot \sin CAP}{\sin OAC \cdot \sin PAB}$ with attention to sign $= \frac{\beta}{\mu} : \frac{\gamma}{\nu}$.

Hence we may replace x, y, z by $\frac{\alpha}{\lambda}, \frac{\beta}{\mu}, \frac{\gamma}{\nu}$ where λ, μ, ν are constants such that $\Sigma\lambda \cdot BC = 0$.

Taking O as the incentre we may put $x = \alpha$, etc.

(2) Corollary.

Project the above figure.

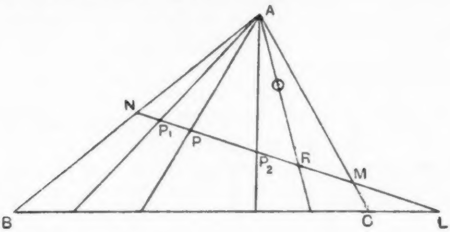
Then the anharmonic coordinates of P are unaltered.

Let the new trilinears be α', etc., λ', etc.

Then $$\frac{\beta}{\mu} = y = \frac{\beta'}{\mu'}.$$

That is the new trilinear coordinates are obtained from the corresponding old trilinears by multiplying each by certain constants.

[These two results were kindly pointed out to me by my former tutor, J. W. Russell, Esq., Balliol College, Oxford.]



IV. *The straight line.*

(1) $P=x, y, z.\quad P_1=x_1$, etc. $P_2=x_2$, etc.

That $lx+my+nz=0$ is the equation of a straight line follows at once from III.

Otherwise $\quad \frac{y}{z}=A(BOCP)=\frac{NR\,.\,MP}{RM\,.\,PN}=\frac{h\,.\,MP}{k\,.\,PN}$, if $\frac{h}{k}=\frac{NR}{RM}$.

Hence by correct choice of ratios we may put

$$\left.\begin{aligned} y&=h(MP)=hp \\ z&=k(PN)=k(n-p)\end{aligned}\right\}$$

taking M as origin and putting $MP=p$, etc.

Now let $$\frac{t}{s}=\frac{P_2P}{PP_1}=\frac{p-p_2}{p_1-p},$$

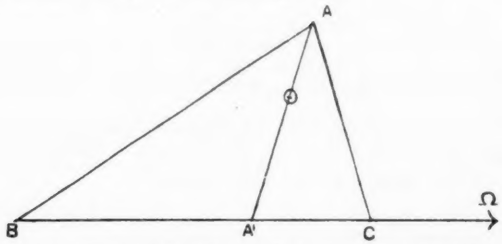
then $$p(t+s)=tp_1+sp_2.$$

Hence $$\frac{y}{z}=\frac{hp(t+s)}{k(n-p)(t+s)}=\frac{thp_1+shp_2}{tk(n-p_1)+sk(n-p_2)}$$

$$=\frac{ty_1+sy_2}{tz_1+sz_2};$$

$$\therefore\ x:y:z=tx_1+sx_2:ty_1+sy_2:tz_1+sz_2. \qquad\qquad (a)$$

Hence l, m, n can be found such that, for all values of $t:s$, $lx+my+nz=0$, which is therefore the equation of a straight line.



(2) A homogeneous equation of the n^{th} degree represents a locus which is cut in n points by a straight line, *i.e.* it is a curve of the n^{th} order.

(3) P_1P_2 is $\|x, y_1, z_2\|=0$.

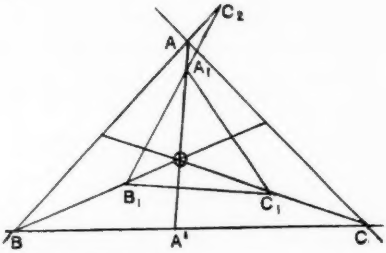
(4) The line at infinity.

Let $$\frac{BA'}{A'C}=\frac{c}{b}, \text{ etc.}$$

Then $$A(BOC\Omega)=\frac{BA'.C\Omega}{A'C.\Omega B}=-\frac{c}{b}.$$

Hence line at infinity is $ax+by+cz=0$.

(5) BC is $x=0$
and AO is $y=z$ $\Big\}$.



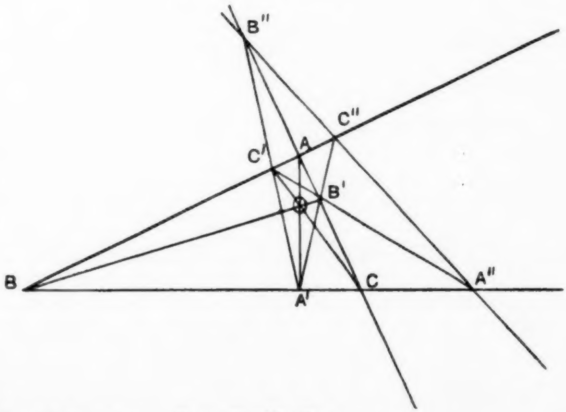
V. (1) Copolar triangles are coaxal.
Let A_1 be $(X, 1, 1)$.
Then A_1B_1 cuts AB where

$$\begin{vmatrix} x & y & 0 \\ X & 1 & 1 \\ 1 & Y & 1 \end{vmatrix}=0$$

or $$x(1-Y)+y(1-X)=0.$$

Hence $A_2B_2C_2$ lie on the line

$$\frac{x}{1-X}+\frac{y}{1-Y}+\frac{z}{1-Z}=0.$$



(2) Let A_1 be A', etc. Then $X=Y=Z=0$.
Hence $A''B''C''$ is the line $x+y+z=0$.

(3) When O is the median centre $a=b=c=1$, and $x+y+z=0$, is the line at infinity (IV. 4).
Hence $A'B' \parallel AB$, etc.

VI. Illustrative example.
Mentioned but not proved by Hamilton.
Unessential to the later sections.
$\Sigma x^2 - 2\Sigma yz=0$ (a) represents a conic touching AB $(z=0)$ where $(x-y)^2=0$, *i.e.* at C', that is, (a) is the inscribed conic touching at A', B', C'.
$\Sigma yz=0$ (b) represents a (reciprocal, IX. (3)) circumscribed conic, for if

$$x=0, \text{ then } y \text{ or } z=0.$$

If P_1, P_2 be the poles of $p \equiv lx+my+nz=0$ w.r. to (a) and (b), we have

$$\left.\begin{aligned}\Sigma(X_1-Y_1-Z_1)x=0\\ \Sigma(Y_2+Z_2)x=0\end{aligned}\right\} \text{ are } p\,;$$

$$\therefore \left.\begin{aligned}\frac{X_1-Y_1-Z_1}{l}&=\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\\ \frac{Y_2+Z_2}{l}&=\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\end{aligned}\right\}$$

$$\therefore \left.\begin{aligned}\frac{Z_1}{l+m}&=\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\\ \frac{Z_2}{l+m-n}&=\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\end{aligned}\right\}$$

But $2(l+m)-(l+m-n)=l+m+n$.
Hence, by IV. (1) (a), OP_1P_2 are collinear.

VII. *The meet line.*
(1) The tangents at A, B, C, to a circumconic, meet the opposite sides on "the meet line."
The general circumconic is $lyz+mzx+nxy=0$.
Tangent at x, y, z is $\Sigma(mZ+nY)x=0$.
Hence tangent at $(1, 0, 0)$ is

which cuts BC where $\left.\begin{aligned}ny+mz=0\\ x=0.\end{aligned}\right\}$

Hence meet line is $\dfrac{x}{l}+\dfrac{y}{m}+\dfrac{z}{n}=0.$

(2) For the circumconic $\Sigma yz=0$.
Meet line is $A''\ B''\ C''$.

F. J. O. Coddington.

(*To be continued.*)

THE INTEGRAL $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

No one can possibly welcome more cordially than I do the widening of the school curriculum in mathematics that has taken place during the last twenty years. Even I can remember the days when 'Conics' and 'Mechanics' were the privilege of a select few, and only a prodigy was initiated into the mysteries of the Differential (much less the Integral) Calculus. All this is changed now, and it is fairly safe (at least that is my experience) to assume that a boy who has won a scholarship has learnt something about the Integral Calculus at school.

I do not believe for a moment that the curriculum has been made *harder*. These happy results are not due to any striking development in the schoolboy himself, either in intelligence, or in capacity or willingness for work. They are merely the result of improvement in the methods of teaching and a wider range of knowledge in the teacher, which enables him to discriminate, in a way his predecessor could not, between what is really difficult and what is not. No one now supposes that it is easy to resolve $\sin x$ into factors and hard to differentiate it!—and even in my time it was the former of these two problems to which one was set first.

Still there are difficulties, even in the Integral Calculus; it is not every schoolboy who can do right everything that Todhunter and Williamson did wrong; and, when I find my pupils handling definite integrals with the assurance of Dr. Hobson or Mr. Bromwich, I must confess I sometimes wonder whether even the school curriculum cannot be made too wide. Seriously, I do not think that schoolboys ought to be taught anything about definite integrals. They really are difficult, almost as difficult as the factors of $\sin x$: and it is to illustrate this point that I have written this note.

I have taken the particular integral

$$(1) \qquad \int_0^\infty \frac{\sin x}{x}\,dx = \tfrac{1}{2}\pi$$

as my text for two reasons. One is that it is about as simple an example as one can find of the definite integral *proper*—that is to say the integral whose value can be expressed in finite terms, although the indefinite integral of the subject of integration cannot be so determined. The other is that two interesting notes by Mr. Berry and Prof. Nanson have been published recently,* in which they discuss a considerable number of different ways of evaluating this particular integral. I propose now to apply a system of *marking* to the different proofs of the equation (1) to which they allude, with a view to rendering our estimate of their relative difficulty more precise.

Practically all methods of evaluating any definite integral depend ultimately upon the inversion of two or more operations of procedure to a limit†—*e.g.* upon the integration of an infinite series or a differentiation under the integral sign: that is to say they involve 'double-limit problems.'‡ The only exceptions to this statement of which I am aware (apart, of course, from cases in which the indefinite integral can be found) are certain cases in which the integral can be calculated directly from its definition as a sum, as can the integrals

$$\int_0^{\frac{1}{2}\pi} \log\sin\theta\,d\theta, \quad \int_0^\pi \log(1-2p\cos\theta+p^2)\,d\theta.$$

Such inversions, then, constitute what we may call the standard difficulty of the problem, and I shall base my system of marking primarily upon them. *For every such inversion involved in the proof I shall assign* 10 *marks.* Besides these marks I shall add, in a more capricious way, marks for artificiality, complexity, etc. The proof obtaining least marks is to be regarded as the simplest and best.

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* A. Berry, *Messenger of Mathematics*, vol. 37, p. 61; E. J. Nanson, *ibid.* p. 113.

† Whether this is true of a proof by 'contour integration' depends logically on what proof of Cauchy's Theorem has been used (see my remarks under 3 below).

‡ For an explanation of the general nature of a 'double-limit problem' see my *Course of Pure Mathematics* (App. 2, p. 420); for examples of the working out of such problems see Mr. Bromwich's *Infinite Series* (*passim*, but especially App. 3, p. 414 *et seq.*).

1. *Mr. Berry's first proof.** This is that expressed by the series of equations

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \lim_{a\to 0}\left(e^{-ax}\frac{\sin x}{x}\right)dx = \lim_{a\to 0}\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx$$

$$= \lim_{a\to 0}\int_0^\infty e^{-ax}dx\int_0^1 \cos tx\,dt = \lim_{a\to 0}\int_0^1 dt\int_0^\infty e^{-ax}\cos tx\,dx$$

$$= \lim_{a\to 0}\int_0^1 \frac{a\,dt}{a^2+t^2} = \lim_{a\to 0}\arctan\left(\frac{1}{a}\right) = \tfrac{1}{2}\pi.$$

This is a difficult proof, theoretically. Each of the first two lines involves, on the face of it, an inversion of limits. In reality each involves more: for integration, with an infinite limit, is in itself an operation of *repeated* procedure to a limit; and we ought really to write

$$\int_0^\infty \lim_{a\to 0} = \lim_{X\to\infty}\int_0^X \lim_{a\to 0} = \lim_{X\to\infty}\lim_{a\to 0}\int_0^X = \lim_{a\to 0}\lim_{X\to\infty}\int_0^X = \lim_{a\to 0}\int_0^\infty.$$

Thus *two* inversions are involved in the first line; and so also in the second. This involves 40 marks. On the other hand this proof is, apart from theoretical difficulties, simple and natural: I do not think it is necessary to add more marks. Thus our total is **40.**

2. *Mr. Berry's second proof.* This is expressed by the equations

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\sum_{i=-\infty}^{\infty}\int_{i\pi}^{(i+1)\pi}\frac{\sin x}{x}\,dx$$

$$= \frac{1}{2}\sum_{-\infty}^{\infty}(-1)^i\int_0^\pi \frac{\sin x}{x-i\pi}\,dx = \frac{1}{2}\int_0^\pi \sin x\sum_{-\infty}^{\infty}\frac{(-1)^i}{x+i\pi}\,dx$$

$$= \frac{1}{2}\int_0^\pi \sin x\,\mathrm{cosec}\,x\,dx = \tfrac{1}{2}\pi.$$

It is obvious that this has a great advantage over the first proof in that only one inversion is involved. On the other hand the uniform convergence of the series to be integrated is, as Mr. Berry remarks, 'not quite obvious.' Moreover the fact that the cosecant series contains the terms

$$\frac{1}{x}-\frac{1}{x-\pi},$$

which become infinite at the limits, although it does not really add to the difficulty of the proof, does involve a slight amount of additional care in its statement. I am inclined to assign 15 marks, therefore, instead of 10 only, for this inversion. For the first three steps in the proof I assign 3 marks each; for the use of the cosecant series 4. Thus the total mark may be estimated at $9+15+4=$ **28.**

3. *Mr. Berry's third proof (by contour integration).* It is naturally a little harder to estimate the difficulties of a proof which depends upon the theory of analytic functions. It seems to me not unreasonable to assign 10 marks for the use of Cauchy's theorem in its simplest form, when we remember that the proof of the theorem which depends upon the formula†

$$\iint\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dx\,dy = \int(p\,dx+q\,dy)$$

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* I mean, of course, the first proof mentioned by Mr. Berry—the proof itself is classical.

† Forsyth, *Theory of Functions*, p. 23.

involves the reduction of a double integral partly by integration with respect to x and partly by integration with respect to y. The actual theoretical difficulty of this proof of Cauchy's theorem I should be disposed to estimate at about 20; but half that amount seems sufficient for the mere use of a well known standard theorem.*

The particular problem of contour integration with which we are concerned is, however, by no means a really simple one. The range of integration is infinite, the pole occurs *on* the contour; we have to use the fact that

$$\lim_{R=\infty}\lim_{\epsilon=0}\left(\int_{-R}^{-\epsilon}+\int_{\epsilon}^{R}\right)\frac{e^{ix}}{x}\,dx=\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx.$$

For this I assign 8 marks; for the evaluation of the limit of the integral round the 'small semicircle' another 8. There remains the proof that the integral round the 'large semicircle' tends to zero. As Mr. Berry points out, the ordinary proof of this is really rather difficult; I estimate its difficulty at 16. Mr. Berry has suggested a simplification of this part of the argument; the difficulty of his proof I estimate at 8. Thus we have $10+8+8+16=\mathbf{42}$ marks for the ordinary proof, and $10+8+8+8=\mathbf{34}$ for Mr. Berry's modification of it.

4. *Prof. Nanson's proof.* This proof, I must confess, does not appeal to me. Starting from the integral

$$u=\int_0^\infty \frac{a\cos cx}{a^2+x^2}\,dx,$$

where a and c are positive, we show, by a repeated integration by parts, and a repeated differentiation under the integral sign, that u satisfies the equation

$$\frac{d^2u}{da^2}=c^2u.$$

As the upper limit is infinite we ought to assign 40 marks for this, following the principles we adopted in marking the first proof; but as the two operations of differentiation involve inversions of the same character I reduce this to 30, adding 6, however, for the repeated integration by parts.

The transformation $x=ay$ shows that u is a function of ac, so that $u=Ae^{ac}+Be^{-ac}$, where A and B are independent of both a or c. This step in the proof I assess at 4.

That $A=0$ is proved by making c tend to ∞ and observing that

$$|u|<\int_0^\infty \frac{adx}{a^2+x^2}=\tfrac{1}{2}\pi.$$

For this I assign 6 marks.

That $c=\frac{1}{2}\pi$ is proved by making $c=0$. Here we assume the continuity of the integral, and this should involve 20 marks. But as the proof is simple I reduce this to 12. Thus by the time that we have proved that

$$u=\tfrac{1}{2}\pi c^{-ca},$$

we have incurred $30+6+4+6+12=58$ marks.

We have then

$$\int_0^\infty \frac{a\sin mx\,dx}{x(a^2+x^2)}=\int_0^\infty \frac{a\,dx}{a^2+x^2}\int_0^m \cos cx\,dc$$

$$=\int_0^m u\,dc=\frac{\pi}{2a}(1-e^{-am}).$$

* Goursat's proof is better and more general, and does not involve my inversion of limit-operations, but its difficulties are far too delicate for beginners.

This involves 20 marks. We then obtain

$$\int_0^\infty \frac{x \sin mx}{a^2+x^2}\, dx = \tfrac{1}{2}\pi e^{-am}$$

by two differentiations with respect to m. This should strictly involve 40, and I cannot reduce the number to less than 30, as the second differentiation is none too easy to justify. The result then follows by multiplying the last formula but one by a and adding it to the second.

I do not think that the $58+20+30=\mathbf{108}$ marks which we have assigned are more than the difficulties of the proof deserve. But of course it is hardly fair to contrast this heavy mark with the 40, 28, 42, 36 which we have obtained for the others; for Prof. Nanson has evaluated the integrals

$$\int_0^\infty \frac{\sin mx}{x(a^2+x^2)}\, dx,\quad \int_0^\infty \frac{\cos mx}{a^2+x^2}\, dx,\quad \int_0^\infty \frac{x \sin mx}{a^2+x^2}\, dx$$

as well as the integral (1).

5. *Mr. Michell's proof.** This depends on the equations

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}\, dx &= \lim_{h \to 0,\, H \to \infty} \int_h^H \frac{\sin x}{x}\, dx\\ &= -\lim_{h \to 0,\, H \to \infty} \int_h^H dx \int_0^{\frac{1}{2}\pi} \frac{d}{dx}\left\{e^{-x \sin\theta} \cos(x\cos\theta)\right\} d\theta\\ &= -\lim_{h \to 0,\, H \to \infty} \int_0^{\frac{1}{2}\pi} d\theta \int_h^H \frac{d}{dx}\left\{e^{-x \sin\theta} \cos(x\cos\theta)\right\} dx\\ &= -\lim_{h \to 0,\, H \to \infty} \int_0^{\frac{1}{2}\pi} \left\{e^{-H \sin\theta} \cos(H\cos\theta) - e^{-h\sin\theta}\cos(h\cos\theta)\right\} d\theta\\ &= -(0-\tfrac{1}{2}\pi) = \tfrac{1}{2}\pi.\end{aligned}$$

I have myself for several years used a proof, in teaching, which is in principle substantially the same as the above, though slightly more simple in details and arrangement, viz.:

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}\, dx &= \frac{1}{2i}\int_0^\infty dx \int_0^\pi e^{i(t+xe^{it})}\, dt\\ &= \frac{1}{2i}\int_0^\pi e^{it} dt \int_0^\infty e^{ixe^{it}}\, dx\\ &= \frac{1}{2}\int_0^\pi dt = \tfrac{1}{2}\pi.\end{aligned}$$

The successive steps of the proof can of course be stated in a form free from i by merely taking the real part of the integrand. To justify the inversion we have to observe that

$$\int_0^\pi e^{it} dt \int_0^X e^{ixe^{it}}\, dx = \int_0^X dx \int_0^\pi e^{it+ixe^{it}}\, dt$$

for any positive value of X, in virtue of the continuity of the subject of integration, and that

$$\begin{aligned}\left|\int_0^\pi e^{it} dt \int_X^\infty e^{ixe^{it}}\, dx\right| &= \left|\int_0^\pi e^{iXe^{it}}\, dt\right| < \int_0^\pi e^{-X\sin t}\, dt\\ &< 2\int_0^{\frac{1}{2}\pi} e^{-X\sin t}\, dt < 2\int_0^{\frac{1}{2}\pi} e^{-2Xt/\pi}\, dt\\ &< \pi(1-e^{-X})/X,\end{aligned}$$

*This proof is also referred to by Prof. Nanson. The only proofs which are 'classical' are 1, 2, 3, viz. the three discussed by Mr. Berry.

the closing piece of argument being essentially the same as that at the end of Prof. Nanson's paper. Hence

$$\int_0^\pi\int_0^\infty = \lim_{X\to\infty}\int_0^\pi\int_0^X = \lim_{X\to\infty}\int_0^X\int_0^\pi = \int_0^\infty\int_0^\pi.$$

This proof involves an inversion of a repeated integral, one limit being infinite. This involves 20 marks. To this I add 15 on account of the artificiality of the initial transformation, deducting 3 on account of the extreme shortness and elegance of the subsequent work. Thus we obtain $20+15-3=$**32** marks. To Mr. Michell's proof, as stated by Prof. Nanson, I should assign a slightly higher mark, say **40**.

We may therefore arrange the proofs according to marks, thus:

1.	Mr. Berry's second proof, - - - - - -	28
2.	Mr. Michell's proof (my form), - - - - -	32
3.	Mr. Berry's third proof (his form), - - - -	34
{4.	Mr. Berry's first proof, - - - - - -	40
{5.	Mr. Michell's proof (his form), - - - - -	40
6.	Mr. Berry's third proof (ordinary form), - -	42
7.	Prof. Nanson's proof, - - - - - - -	108

And this fairly represents my opinion of their respective merits: possibly,* however, I have penalized Nos. 2 and 5 too little on the score of artificiality and 4 too much on the score of theoretical difficulty. I conclude, however, with some confidence that Mr. Berry's second proof is distinctly the best. But whether this be so or not, one thing at any rate should be clear from this discussion: whatever method be chosen, the evaluation of the integral (1) is a problem of very considerable difficulty. This integral (and all the other standard definite integrals) lie quite outside the legitimate range of school mathematics.

G. H. HARDY.

REVIEWS.

The Foundations of Mathematics: A Contribution to the Philosophy of Geometry. By DR. PAUL CARUS. Chicago: The Open Court Publishing Company. 1908. Pp. 141.

This book is a more or less popular exposition of a philosophy of geometry which is, in its main outlines, derived from Kant. The main title, if uncorrected by the sub-title, would be somewhat misleading, since the foundations of arithmetic and analysis are not discussed, but only the foundations of geometry. The author begins by a brief account of the development of non-Euclidean Geometry, which is followed by much longer chapters "on the philosophical basis of mathematics" and on "mathematics and metageometry." The historical chapter, though it does not profess to give more than a sketch, might with advan-

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*This I know to be Mr. Berry's opinion. He would put 2 below 3 and 4, and 5 below 6. To compare 3 and 4 is very difficult, and they might fairly be bracketed.

Since writing this note I have recollected another proof which I had forgotten, and which is given in § 173 (Ex. 1) of Mr. Bromwich's *Infinite Series*. Mr. Bromwich there proves that, provided the real parts of a and b are positive or zero,

$$\int_0^\infty (e^{-ax}-e^{-bx})\frac{dx}{x} = \int_0^\infty \left(\frac{1}{a+y}-\frac{1}{b+y}\right)dy.$$

Putting $a=1$, $b=i$, we obtain

$$\int_0^\infty (e^{-x}-\cos x)\frac{dx}{x}=0,\quad \int_0^\infty \frac{\sin x}{x}dx=\tfrac{1}{2}\pi.$$

This proof, as presented by Mr. Bromwich, should be marked at about **45**. It has the advantage of giving the values of several other interesting integrals as well, so that (as in the case of Prof. Nanson's proof) it is hardly fair to contrast this mark with the considerably lower marks obtained by some of the other proofs.

tage have been enlarged by some account of projective geometry and the projective treatment of metrics. Dr. Carus speaks always as though non-Euclidean straight lines were not really straight, but were merely called straight out of wilfulness. The projective treatment shows, better than the metrical, wherein the straight lines of non-Euclidean spaces agree with those of Euclid, and ought therefore not to be omitted even in a mere outline. It would seem also that Dr. Carus regards a three-dimensional non-Euclidean space as necessarily contained in a four-dimensional Euclidean space, for he asks "what Riemann would call that something which lies outside of his spherical space," apparently not realizing that spherical space does not require anything outside it.

The author's philosophical theory of geometry may be briefly summarized as follows. Geometry, like logic and arithmetic, is *a priori*, but it is not *a priori* in the same degree as logic and arithmetic. There is the *a priori* of being and the *a priori* of doing, and geometry belongs to the latter: it is derived from the contemplation of motion, and can be constructed from the "principles of reasoning and the privilege of moving about." We know *a priori* what are the possibilities of motion; thus, although there is nothing *logically* impossible about the assumption of four dimensions, yet "as soon as we make an *a priori* construction of the scope of our mobility, we find out the incompatibility of the whole scheme." The *a priori* is identical with the purely formal, which originates in our minds by abstraction; it is applicable to the objective world because the materials of formal thought are abstracted from the objective world.

Most of the arguments in the book lead one to expect that Euclid will be declared to be certainly alone valid as against non-Euclidean geometry, yet this is not the conclusion drawn by the author. He says: "The result of our investigation is quite conservative. It re-establishes the apriority of mathematical space, yet in doing so it justifies the method of metaphysicians in their constructions of the several non-Euclidean systems. . . . The question is not, 'Is real space that of Euclid or of Riemann, of Lobatchewsky or Bolyai?' for real space is simply the juxtaposition of things, while our geometries are ideal schemes, mental constructions of models for space measurement. The real question is, 'Which system is the most convenient to determine the juxtaposition of things?'" (p. 121). Yet a few pages later he says: "The theorem of parallels is only a side issue of the implications of the straight line" (p. 129). It is not clear how these statements are reconciled, for the earlier statement seems to imply that there is no "theorem" of parallels at all.

A few of the author's assertions are somewhat misleading. For example, he states, as a fact not open to controversy, that Euclid's axiom or postulate of parallels originally occurred first in the proof of the twenty-ninth proposition, not being mentioned either among the axioms or among the postulates (p. 2). On the other hand, Stäckel and Engel (*Theorie der Parallellinien*, p. 4) say that, following Heiberg, they do not regard the postulate of parallels as a later addition, which would seem to show that Dr. Carus's opinion is at least open to question. Again he says (p. 84): "While in spherical space several shortest lines are possible, in pseudospherical space we can draw one shortest line only." As regards spherical space, the more exact statement is that in general only one shortest line can be drawn between two given points, but when the two points are antipodes, an infinite number of shortest lines can be drawn between them.

The book concludes with an epilogue, in which the existence and attributes of the Deity are deduced from the nature of mathematical truth.

B. Russell.

"A Modern Zeno." Mr. Francis C. Russell, in the *Monist* for April, pp. 289-308, gives his reasons for thinking that there is an analogy between Lobatchewsky's non-Euclidean geometry and Zeno's paradox of Achilles and the tortoise. It seems sufficient to quote his belief (p. 298) that the law of contradiction, though it is "in its subjective application insuperable, at least in our present state of intellectual development," is not necessary in its "objective application" to continue. I have heard also of a domestic who doubted the application of "A is either out or not out to the real world."

On p. 303, Mr. F. C. Russell expresses a wish "that mathematicians would have less contempt for the philosophers"—a wish that cannot but be generally shared. He goes on: "In Dr. Paul Carus we have an instance all too rare of a

philosopher fully up to date and fully furnished in mathematics so far as the same has any philosophical import. His *Foundations of Mathematics* deals with the questions now most in gestation in a way and with a mastery that mathematicians can ill afford to neglect." A review of Dr. Carus' book appears above this notice.

PHILIP E. B. JOURDAIN.

A Study of Mathematical Education, including the Teaching of Arithmetic. By BENCHARA BRANFORD, M.A., Divisional Inspector to the London County Council, formerly Lecturer in Mathematics in the Victoria University, Oxford. Pp. xii + 392. 1908. (Clarendon Press.)

I have seldom read a book with the motives of which I feel in such sympathy, and with the conclusions of which I so fully agree as this one. Mr. Branford's work is an advocacy, based on a large and varied experience of teaching, of heuristic and historical (see especially pp. 256-262) methods in mathematical education. All candid inquirers, surely, would admit the truth of the principle (p. 244) that "the path of most effective development of knowledge and power in the individual, coincides, in broad outline, with the path historically traversed by the race in developing that particular kind of knowledge and power." And the teacher has to conduct his experiments on the broad lines suggested by the racial development of mathematical experience; in fact, Mr. Branford re-translates the fundamental laws of this racial development into terms of individual development.

The experiments in the teaching of elementary geometry to normal (chs. i.-iv.) and blind (ch. ix.) children, the discussion of the teaching of arithmetic (ch. v.-vii.), the illustration of the types of proof suitable for beginners in geometry (ch. viii.), the discussion of sub-conscious experience (ch. x.), and the brief notes of lessons (ch. xiv.) are extremely valuable contributions to the psychology of teaching, and continually illustrate the presence of what may almost be described as a natural genius for mathematics in children which is so often crushed by artificial and stupid methods miscalled "educative." No doubt these methods arose from a very noble straining after logical perfection—or as much logical perfection as the learners were thought to be capable of—and no doubt, as Mr. Branford puts it (p. viii), "the particular type of intellectual discipline obtainable from mathematical study on its formal, systematic, and logical side, is in considerable danger of becoming temporarily sacrificed during a too extreme swing of the pendulum of reform." But it is an unfortunate fact that, while the "improvements" of text-books on historical expositions are, in almost every case, ludicrously inadequate when judged according to strict logic, they have often introduced errors of a kind that a child would instinctively reject, that a modern logician would reject with consciousness of the nature of the error, and which would be and have been eagerly accepted by those whose natural intelligences have been stunted by those very text-books. I have tried to illustrate the latter point* in this *Gazette*† by pointing out how even eminent men like Heine, Helmholtz, Pringsheim, and Stolz have made the confusion—so common among mathematicians—between sign and signification. This confusion, it seems to me, is entirely due to the abandonment or disregard by some teaching of historical, living methods, and its consequent degeneration into a lifeless formalism.

This of course raises too large a question to be treated here. The haphazard use of the vocabulary of the significate and its value tends to atrophy the normal child's clear if naïve view that all experience is sign, and that all signifies: an instinctive outlook which, until the impulse is balked or dies down, prompts him to bombard his elders with questions that to them seem frivolous or unanswerable. This remark is made by Lady Welby.

Between the natural method of teaching in which our instincts are trained—advocated in Mr. Branford's book—and the manner in which modern logic treats mathematical questions, there is a third method still pursued in many schools, whose object is very like what Mr. F. H. Bradley, in the preface to his *Appearance and Reality*, humorously stated to be the object of Metaphysics: "The

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* The former point has frequently been shown in reviews by several writers in the *Revista di Matematica*.

† Vol. IV., pp. 207, 208 (Jan., 1908, Part II.); cf. Vol. IV., p. 307 (July, 1908).

finding of bad reasons for what we believe upon instinct." This third method is superfluous; it corresponds to no stage which the human mind has passed when starting from the first method and following an ideal it has reached the second method and point of view. And there is no gap between the first and second methods, for the first merges insensibly into the second.

A last remark that must be made on Mr. Branford's book is that there are useful accounts in it of the development of notation in arithmetic and algebra, and, in this connection, the "principle of the economy of thought" is rightly emphasised. PHILIP E. B. JOURDAIN.

Plane Geometry for Advanced Students. Part I. By C. V. DURELL, M.A. Pp. xi, 219. 1909. (Macmillan & Co.)

This volume has been written in the first instance to supply the wants of Winchester College, and it deserves to be widely known; we wish it every success, and hope that it will become generally adopted, for it is a good book. Instead of a mass of bewildering details without method or connecting thread, such as we have become accustomed to expect in English text-books on the earlier parts of Geometry, we are given the essence of what is useful for further progress. The last five pages of the text seem somehow to have got stranded, but with this exception, everything falls into its natural place, forming an interesting and connected whole. The book is singularly successful in bringing out the principles of Geometry; and it forces the reader to think, although it is seldom hard to understand. It is also remarkably complete, nothing essential being omitted, and very little that is unessential being admitted.

The book may be described as an immediate and complete sequel to Euclid. The first four chapters (covering fifty pages) fill up the lacunae of Euclid; and then follow chapters on Concurrency and Collinearity, Vector Geometry, Mean Centre, Harmonic Ranges and Pencils, Principle of Duality, Inversion, Poles and Polars, and Coaxal Circles. All of these are excellent and thorough, and the chapters on Vector Geometry, Principle of Duality, and Inversion contain a considerable amount of important matter which is not to be found in other sequels to Euclid.

Most of the chapters are prefaced by short historical notes, which add something to their interest. We only make one small criticism. The author, with others, makes *sense* a useless synonym for *direction*, and, by defining one in terms of the other, leaves both in reality undefined. Sense is an elementary notion, which should be applied to a single magnitude only, and serves to distinguish between forward and backward movement; to give it an extended meaning is only to introduce confusion. Direction, on the other hand, is an abstruse and distinctively geometrical notion.

We shall look forward with interest to Part II., which is to include the properties of conics. If we may express a personal opinion, we would suggest that as Part II. will be more advanced, so it ought to be more condensed, while retaining the same degree of exactness. Part I. seems to us to be perfectly adapted to its purpose; but the reader of Part II. will be equipped with greater knowledge, and more should be left to his reasoning powers. It is only by giving the student gradually stronger meat that he can attain to his full development. Again, would it not be better both for the writer and the reader to have fewer examples? Part I. contains about four times too many; and a misguided youth may attack hundreds of easy questions, and never a hard one, without discovering his folly. The English system of cramming a book with examples is simply vicious, and a positive hindrance to mental progress. F. S. MACAULAY.

Das Prinzip der Erhaltung der Energie. By DR. MAX PLANCK. 2nd Edition. Pp. xvi, 278. 1908. (Teubner.)

The book before us forms the sixth of a series published by Teubner & Co., under the title *Wissenschaft und Hypothese*, Professor Poincaré's well-known book with this title giving the name to the series, and taking the first place.

Professor Planck's work is the second edition of a book of the same title, written as a Prize Essay of the University of Göttingen, and published in 1887.

To give a brief outline of the contents of the book, which divides itself naturally into three sections : in the first part the historical development of the subject is treated very fully and in a very interesting manner; in the second part the precise definition of the principle of the conservation of energy is given, together with a review and criticism of the various proofs of it which have been submitted; the last section deals with the different kinds of energy that arise in Physics, and shows the way in which the particular problems that arise can be attacked by means of the principle of its conservation. In the new edition the author has not deemed it necessary to make many alterations, limiting himself to the addition of a few footnotes, where later research has led him to modify the original statements.

An attempt to include even a small amount of the research of the last twenty years in Thermodynamics and Electrodynamics, where the conceptions of "Energy" and the "Conservation of Energy" are of fundamental importance, would have completely altered the original character of the book, besides adding greatly to its size. In view of the different ideas put forward concerning the Principle of the Conservation of Energy, the author's view, given in a footnote on p. 111, may be quoted : "Daraus folgt, dass das Energieprinzip weder eine Tautologie ist, noch eine verkleidete Definition, noch ein Postulat, noch ein Urteil à priori, sondern ein Erfahrungssatz." Whilst we feel with the author, that to the modern physicist the book will present many gaps, we can recommend it as a very full account of the conception of Energy, the historical treatment used adding greatly to the interest of the book.

H. R. HASSÉ.

Joannis Verneri de Triangulis Sphaericis Libri quatuor. Herausgegeben von AXEL ANTHON BJORNBO. (Teubner.)

We have here Part I. of a work of Werner's, hitherto existing only in MSS., a copy of which has been recently found by persevering investigators in the Vatican library. The remaining MSS. comprise *De Meteoroscopiis* Libri Sex. We congratulate the editor and all concerned in the publication of this work on the successful issue of their undertaking. It is a highly interesting contribution to the history of Spherical Trigonometry. Though in ordinary historical sketches of Mathematics and Astronomy not much is said of Werner, his name is, as far as our experience goes, always mentioned with honour. Thus Baden-Powell speaks of his acquaintance with the writings of the ancient geometers at a time when these were difficult of access, while Galloway and others attribute to him the first explanation of the method of obtaining longitude by Lunar Distances. Montucla, indeed, singles him out for special praise, as "peu connu et qui méritait de l'être d'avantage," not merely as "un habile géomètre;" but also "un astronome de grande réputation qui marche de près sur les traces de Purbach et de Regiomontanus." Born at Nuremburg in 1468, and educated in its schools, he betook himself at the age of twenty-five to Rome, where he seems, from passages in his works, to have remained until 1497, about the end of which year he returned to his native city. Here he passed the rest of his life as a priest. He died in 1528. Such time as his ecclesiastical duties left at his own disposal he devoted to mathematics and astronomy as a keen student and diligent observer. He projected, as many other enthusiasts have, more works than he ever brought to a successful termination, and, in particular, the treatise on 'Spherical Triangles' here before us, which, unfinished as it is, would have brought him additional fame had it not, by a fate which the editor fitly describes as "recht trauriges," remained unpublished until long after his discoveries had been remade or utilised without acknowledgment by others. Its publication seems to leave no doubt that he was the real inventor of the 'Prosthaphæretic' method of evading the tedious multiplication of the sines and cosines of observed angles by means of such formulae as

$$2\sin A \sin B = \cos(A-B) - \cos(A+B),$$

and the first to obtain an equivalent to the fundamental formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Of the four books, I. is devoted to properties of spherical triangles without reference to trigonometrical formulae;* II. gives some general theorems, including that of Menelaus of which he gives an extremely neat demonstration.

III. Shows how to solve oblique-angled triangles by decomposition into right-angled ones. IV. illustrates the use of the fundamental theorem quoted above expressed in its 'prosthaphæretic' form.

$$\frac{\frac{1}{2}\{\sin(90^\circ - a + c) - \sin(90^\circ - c - a)\}}{\sin(90^\circ - b) - \sin(90^\circ - c - a)} = \frac{r}{\sin \operatorname{vers}(180^\circ - B)},$$

the ungainly appearance of the formula arising from the fact that he makes no use of the *cosine*, but merely of the '*sinus rectus*' (our *sine*) and the '*sinus versus*' (our *vers'ed sine*). Though the advantage of 'Prosthaphaeresis' declined on the introduction of logarithms, its invention was a great step at the time it was made. Tycho Brahé and his assistant Wittich, who seem to have used it in their calculations, are sometimes regarded as its inventors. Montucla in the first edition of his *Histoire* attributes it to Raimard Ursus;† but in the second he expressly states (p. 584), that "il faut remonter encore plus haut pour retrouver l'origine de cette méthode prosthaphérétique. Car nous apprenons de Christmann dans son *Theoria Lunae* (chap. 17), que son premier inventeur fut le chanoine Werner de Nuremberg," and he considers it of such interest that he devotes a note of considerable length (B. pp. 617-619) to show the actual and possible uses of it.‡ On the analogy of prosthaphaeresis and logarithms when analytically considered, restricted, indeed, but more than superficial, see Glaisher's article on Napier (*Encycl. Brit.*). Napier himself gives some examples of its use in an appendix to his *Constructio*, and we have found the word in use as late as the time of Abraham Sharpe, who in a letter to Flamsteed (dated May 28, 1706), says: "I have no leisure to make any further progress in the tables of *prostaphœreses*"; but we are uncertain whether the term is here used in its ordinary trigonometrical sense or for something logarithmic. The *De Meteoroscopiis* Lib. VI. are not, as might be rashly guessed, concerned with weather prediction, but with instruments for solving astronomical problems such as an armillary sphere or a planisphere. Many such appear to have been invented and described before the time arrived when improved methods of trigonometrical and logarithmic calculations, and the publication of copious tables, rendered them unnecessary for an astronomer's outfit. We are not at all sure, however, that there is not still a use for them in elementary studies, especially now that graphic methods are so much in vogue.

We have noticed with interest that in the announcement by Philip & Son of the forthcoming *édition de luxe* of Ravenstein's 'Martin Behaim, his Life and his Globe,' we are promised descriptions of several of these instruments.

E. M. LANGLEY.

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*Though it is generally stated that the use of sines was not substituted for that of chords until the middle ages, it may be noticed that Menelaus himself, by the use of the 'chord of double the arc' (or '*nadir* of the arc' as he called it) was enabled to extend his theorem on the transversal of a plane triangle to that of a spherical triangle, almost as neatly as Werner. Neither put it into the modern form. The student now generally meets the property of the plane triangle some time before his introduction to spherical trigonometry, whose needs led to its invention, and the analogous theorem for spherical triangles gets an independent proof. But Werner's proof (possibly following that of Menelaus) is worth notice. Project the spherical triangle ABC into the plane triangle ABC, and the traversal arc DEF into the transversal straight line def. Then by the plane theorem,

$$Af \cdot Bd \cdot Ce = fB \cdot dC \cdot eA.$$

But, $$Bd/dC = \sin BD/\sin Dc, \text{ etc.};$$

$$\therefore \sin AF \cdot \sin BD \cdot \sin CE = \sin FB \sin DC \sin EA.$$

†Montucla's form for Reymers Bär. [Ed.]

‡There seems to be an error in the editor's reference to Montucla's *Histoire des Mathématiques*, I. Paris, 1758. It should be to the second edition (1799-1802).

Analytische Geometrie auf der Kugel. Von Prof. Richard Heger. (Leipzig, G. J. Göschen.)

A compact treatise of about 150 pages, in which the properties of points and lines on the sphere are treated by means of various systems of spherical co-ordinates. Thus, we have those invented by Gudermann (according to De Morgan independently by Graves), in which ABC being a quadrantal triangle the position of P is fixed by taking C as origin and as co-ordinates $\tan CE=x$ and $\tan CD=y$, APD, BPE being quadrantal arcs through P. If the radius of the sphere be taken as unity these are of course Cartesian co-ordinates of the gnomonic projection of P on the tangent plane at C. De Morgan proposed to call these 'Tangential,' considering the term 'spherical' as already too closely identified with systems of co-ordinates used in Astronomy. Graves' term 'spherical' has, however, prevailed, and is used to cover other systems which have since been invented. Thus Gudermann afterwards chose a still neater system, based on the same quadrantal triangle ABC, drawing in the quadrantal arc CPF, and taking $x=\sin DP$, $y=\sin EP$, $z=\sin FP$. Since, $x^2+y^2+z^2=1$, we are by this choice sometimes enabled to render an equation homogeneous and use the analogies of trilinear co-ordinates. Gudermann called these 'Distanz' co-ordinates: with the radius of the system unity (x, y) are Cartesian co-ordinates of the orthogonal projection of P on the tangent plane at A. Dr. Heger mainly employs this system, though he makes some use of the 'Tangential,' and points out that the position of P with reference to the triangle ABC may be fixed by means of co-ordinates analogous to the barycentric co-ordinates of Möbius by the consideration of the forces (L, M, N), which, acting along OA, OB, OC, would give a resultant along OP, when O is the centre of the sphere. After discussing various noteworthy points of the spherical triangle, the author proceeds to the investigation of sphero-conics, to the properties of which about half the book is devoted. This part may be read with advantage either before or concurrently with the sections of Reye, dealing with the same subject from a more purely geometrical point of view. Such a book was much needed. Attention was drawn to these curves in the latter editions of Salmon's *Solid Geometry* and in Robert's *Examples in Analytical Geometry.* A list of some of their most striking analogies with the corresponding plane curves was added by Routh to his *Rigid Dynamics*, with the remark that they appear to be new; but we know of no text-book of moderate size in which a student can find a fairly complete account of them, such as is here given; and the works of the forty or so writers mentioned by Dr. Heger as having treated of them during the last century are not generally accessible to students. Readers of the *Gazette* may perhaps remember a valuable note by Prof. F. Morley (Vol. I., p. 249), in which the well-known figure of the solid cone common in text-books on Geometrical Conics is utilised for the sphero-conic. This can be further utilised by those who prefer synthetic treatment. Some theorems on spherical cubics are added analogous to those given by Maclaurin and Chasles for the plane. E.g. *If a spherical cubic has three real points of inflexion they all lie on a great circle.* We notice a trifling error in the spelling of *Quarterly*, p. vii, l. 19, repeated on p. 42. The book would be much improved by the insertion of more diagrams, especially by one like Routh's, accompanied like it by a list of properties enunciated briefly in the following way: $PL=$lat. rect.; $\tan PG \,.\, \tan PF=\tan^2 b$.

E. M. Langley.

On Models of Cubic Surfaces. By W. H. Blyth, M.A. (Camb. Univ. Press.)

In the author's words, "the object is to give an outline of analytical and geometrical methods that are used in treating of cubic surfaces, not taking the more advanced part of the subject, but considering mainly anything that may help to the construction of models." While welcoming this little book as one of a sort of which in England we get too few, we are inclined to think it is either too big or not big enough. The introduction, which may perhaps serve well enough as a reminder to one who has known his Salmon's *Solid* from cover to cover, is rather too concise for those who have never been in that happy condition, and, lucidly as the results are stated in it, we do not think it is easy enough to lure on those who are apt to be deterred by initial difficulties, though capable

of sustained endeavour when they once feel that they have made a beginning. The same remarks apply to chapters I. and II. We think the whole would be much improved by the introduction of diagrams illustrating the various definitions and theorems, especially photographs of actual models or parts of such. It may be that the class of readers the author had in his mind would not want such aids; but for these surely the introduction and the chapters we have referred to need not have been written at all, for references to Salmon, Frost, and Reye, and a mere list of the theorems to be used would have been enough. When once, however, the author gets to his models we have no reason to complain of any deficiency in the matter of explanatory diagrams, which seem well and clearly executed. The book should stimulate interest in a subject well calculated to give absorbing occupation to those fond of construction, and with leisure enough to enable them to devote a fair amount of time to it. It will probably do a subsidiary benefit in drawing attention to the powerful geometrical methods expounded in the *Geometrie der Lage*. We see no mention of Wiener's name, who in 1869 published an interesting little pamphlet descriptive of the finished model begun by him in the winter of 1867, in response to a request made by members attending the mathematical section of the Scientific Association at Frankfort in the previous September. The pamphlet was brought out to accompany and explain two nicely executed stereoscopic slides of the surface in question. These slides could, until quite recently, and for anything we know can perhaps still, be obtained (B. G. Teubner, Leipzig), and those interested in such matters should lose no time in getting them. We presume that this surface is the subject of Cayley's lengthy memoir (*Camb. Phil. Trans.*, May 15th, 1871).

E. M. LANGLEY.

Vorlesungen über Differential Geometrie. By R. V. LILIENTHAL, Bd. I. Pp. 368. (Leipzig: Druck und Verlag von B. G. Teubner, 1908.)

This new work on differential geometry is confined as far as the first volume is concerned to the theory of curves. Plane curves are first dealt with, then space curves. The point of view from which the former are examined is much more advanced than is generally found in text-books on the subject. Points on a curve are expressed by co-ordinates in the form

$$x=f_1(t),\quad y=f_2(t),$$

and the aim is to develop the usual differential theory of the curve in such a way that the cases of ordinary and singular points are included in one. This is done as follows: Corresponding to small increase Δt in t there are small increases Δx, Δy in x, y. If these are represented by power series such as

$$\Delta x=\Sigma\frac{1}{n!}f_1^{(n)}(t)\Delta t^n,\text{ and a similar expression for }\Delta y,$$

it is assumed that ν is the lowest index of Δt which does not vanish, and thus the direction cosine $\cos\alpha$ of the tangent is expressed as

$$\operatorname{Lt}\frac{\Delta x}{\Delta t}=\frac{\epsilon^\nu f_1^{\nu}(t)}{W_\nu(t)},\text{ where }\epsilon\text{ is}\pm 1\text{ and }W=\sqrt{(f_1^{\nu})^2+(f_2^{\nu})^2}.$$

Again, writing the equations of the tangent and normal as

$$u=(\xi-x)\cos\alpha+(\eta-y)\sin\alpha,$$
$$v=-(\xi-x)\sin\alpha+(\eta-y)\cos\alpha,$$

v may be similarly developed as a power series in Δt of which the lowest index is $\nu+\lambda$. Upon the oddness or evenness of ν and λ depends the shape of the singularity, if one exist at the point (t). For an ordinary point ν would be unity.

Upon these lines the curvature, evolute, involutes, etc., of a curve are discussed. The greater part, however, of this section of the volume is devoted to systems of curves. Firstly, taking

$$x=f_1(t,\tau),$$
$$y=f_2(t,\tau),$$

to represent a system of curves where τ is the parameter of the system, the theory of envelopes is dealt with. Subsequently these equations are used as defining two systems of curves, t and τ being on the same footing. The theory of parallel curves, isometric curves, and of doubly infinite sets of circles are

touched upon in this chapter. The section ends with a short reference to Lie's theory of infinitesimal transformations.

Curves in space are approached in the usual analytical way of working out the principal directions and curvatures in terms of x, y, z and an independent variable t. There follows an unusual section dealing with singular points in the same way as that in which those for plane curves were met. Needless to say the analysis is very heavy and complicated. The rest of the book deals with a wider field of curve geometry:—the relation of the linear complex to curves, orthogonal trajectories, evolutes, and some particular classes of curves, such as Bertrand's type, where the curvature and torsion are connected by a linear relation. Lastly, the kinematical view of the theory is developed at some length, in which screw motion and the theory of linear complexes are both worked out. The motions of lines and planes, with reference to the curves described by their constituent points, are investigated in this connection.

Altogether the book contains as full and complete account of the differential theory of curves as exists at the present time. H. W. TURNBULL, B.A.

Analytische Geometrie der Ebene. By C. RUNGE, Professor at Göttingen University. Pp. 198. (Leipzig und Berlin: Druck und Verlag von B. C. Teubner, 1908.)

This work has been compiled for the benefit of technical students by an author who has treated the subject in lecture at the Technical High School, Hanover. It is, therefore, not an exhaustive treatment from a geometrical point of view, though it is very thorough as far as it goes. The range covered is the analytical geometry of straight lines, circles, and conic sections, omitting the theory of confocal conics and kindred subjects usually included in a text-book on conics. The theory of vectors is clearly and practically set forth, followed by a detailed discussion of equations of the first order. In particular there is an interesting section dealing with the application of this geometry to the statics of frameworks of light rods. The geometry of the circle is discussed very fully, with particular reference to the connection between the solution of quadratic equations and problems involving circle and line constructions. An interesting feature of this section of the book is the use made of the 'power' of a point with regard to a circle, especially in bringing out the chief properties of coaxial circles.

The theory of conic sections is developed from the circle by the introduction of projection and transformation of coordinates. Orthogonal projection naturally leads to the ellipse, and all the principal metrical properties are deduced. By employing suitable simple linear transformations of coordinates all the different types of conics are deduced from the circle, and finally it is pointed out that a composition of these transformations is equivalent to the general linear transformation. This naturally leads to a discussion of the general equation of the second degree and to the use of homogeneous coordinates. The equivalence of the theory of linear transformations in analysis and projection in geometry is explained better and proved more rigidly than is usual in text-books on analytical geometry. H. W. TURNBULL, B.A.

A Course of Mathematics for Students of Engineering. By F. S. WOODS and F. H. BAILEY, Professors of Mathematics in the Massachusetts Institute of Technology. Vol. I. Pp. xii, 385. (Ginn & Co.) Price 10s. 6d.

This volume deals with algebraic equations, analytical geometry and the differential calculus.

The accurately drawn curves are a distinguishing and valuable feature of the book. There is quite as much character for instance in the cubic $y^2=x.(x-1)^2$, when accurately plotted with a slide rule, as there is in an ellipse; and just as much is lost by a perfunctory sketch of the cubic, as by drawing an oval of uncertain antecedents instead of the graceful ellipse. The authors have succeeded very happily in their aim of maintaining a reasonably high standard of mathematical precision with due regard to the requirements and capacities of the students for whom the book is meant, and they have provided an abundant and admirable selection of illustrative problems and exercises. The topic to which every teacher will at once turn in a book like the present is the treatment of the exponential function, but he will find that the subject is in part postponed to

the second volume, and that the not very satisfactory method is adopted which might be styled daemonic, or named after Wemmick.

Hullo! Here's a series

$$1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots.$$

Let's have a look at it!

But it is hardly fair to complain that the authors have not achieved what seems to be impossible—a simple, natural, and reasonably sound treatment of the exponential function. C. S. JACKSON.

Les Origines de la Statique. By P. DUHEM. Vol. I. Pp. iv, 360. Vol. II. Pp. viii, 364. 1905, 1906. (Hermann.)

Those who were privileged to read many of the chapters of which these two fine volumes are composed at their first appearance in the pages of the *Revue des Questions Scientifiques* will rejoice at their republication in book form. Five years ago the author published his monograph on the *Evolution of Mechanics.* The number of books on the history of Statics is very small indeed, and although none of them goes much into details, an ordinary reader would feel himself entitled to suppose that at any rate they may be depended upon as far as the broad lines of the subject are concerned. But Prof. Duhem is not an ordinary reader. Those who are familiar with his works on Mathematical and Chemical Physics, on Physical Theory, with his articles on philosophical subjects, and on the origin and growth of ideas, know him to be an erudite scholar of prodigious industry. To many of our readers whose knowledge of the history of Statics will have been derived in the main from the *Encyclopaedia Britannica*, or Mach's book on Mechanics, and similar sources, the discoveries of Prof. Duhem will come as a revelation. To put the situation in a nutshell, if the metaphor be permissible, the author had occasion to test the statements made in existing histories by reference to the original sources. The unexpected happened, as it sometimes will. "L'ensemble même de l'histoire de la Statique pût être bouleversé par nos recherches." He found that so far from the views on matters mechanical held by Leonardo de Vinci being unknown to the mathematicians of the Renaissance, they had been exploited by many during the sixteenth century, especially by Cardan and Benedetti. Upon them Cardan, in particular, had drawn in forming his ideas on the motive power of machines and on the impossibility of perpetual motion. More surprises were to come. He read Tartaglia, whose name he declares is not to be found in more than one (G. Vailati) of the writers he consulted. He would have found an account of him and a sketch of his works in Rouse Ball and in the *Encyclopaedia*, though here his chief claim to fame is attributed to his merits as an algebraist. Even the omniscient De Morgan, in the *Penny Encyclopaedia*, was not aware that Nicholas the Stammerer claimed to have discovered the apparent weight of a body on an inclined plane. But, as was pointed out at the time by Ferrari, this was an impudent plagiarism on the part of Tartaglia, who had stolen it from a geometer of the 13th century—Jordanus Nemorarius. Now of Jordanus two treatises had been published in the 16th century, clearly not the work of the same hand, many of the positions held being irreconcilable. So Prof. Duhem made a raid on contemporary manuscripts in the Sorbonne and other great libraries, and again the unexpected happened. Not only had the West received Greek traditions as to the doctrine of the lever and the Roman balance, either directly or through the Arabs, but there had grown up an autonomous Statics unsuspected by antiquity. From the early years of the 13th century Jordanus had proved the law of the lever, starting from the postulate that the same power will lift different weights if those weights are inversely as the distances through which they are raised. The germ found in the work of Jordanus developed rapidly, and is to be found in succession in the writings of his own disciples, Leonardo de Vinci, Cardan, Roberval, Descartes, and Wallis. We see it in its final form in John Bernoulli's letter to Varignon, in the *Mécanique Analytique* of Lagrange, and finally in the work of Willard Gibbs. So that the science of which we are so proud was derived by a gradual evolution from the germ sown about the year 1200. And the author goes so far as to claim that he has shown that in both mechanics and physics there

has been at no stage an intellectual revolution, and that respect for tradition is an essential condition for all scientifique progress.

As far as we are competent to form an opinion, Prof. Duhem seems to have made out a *primâ facie* case as far as the science of mechanics is concerned. We understand that he is still pursuing his researches in the great libraries, and will perhaps in the future be able to fill the few lacunae over which he is at present perforce carried by (not unwarrantable) assumptions at various stages of his argument. Another name emerges from the obscurity of the 14th century in the person of Albert de Saxe, or Albertutius, better known to the scholastics as Saxonius. He is to be found neither in Mach nor in Mr. Rouse Ball's *History*. The germ of the doctrine that the world is motionless at the centre of the Universe is to be found in Aristotle. This was revived and expanded by Albert de Saxe, and may thus be briefly stated:—In every heavy body there is a point at which the whole weight is concentrated—the centre of gravity. The centre of gravity of every body craves to coincide with the centre of the Universe. If it is there already the body is at rest. If it is not there, it moves towards it, if not prevented, in a straight line. The Earth is a heavy body like those of which we wot, and it has joined its centre of gravity to that of the Universe. This doctrine is of sufficient importance to be placed on record. It went a long way towards giving the right direction to the revolution initiated by Copernicus, and it is a starting-point for a sounder theory of the centre of gravity which finally appears with Galileo, Torricelli, and Kepler. Another interesting section of M. Duhem's delightful volumes is devoted to an account of the remarkable blunders made by Fermat in his famous discussions with Roberval, Pascal, and Descartes. But space fails, and we must make but one further observation. In the account given by the author of the shaping of the parallelogram of forces in the hands of Varignon, Roberval, etc., we find no mention of Duchayla, whose proof, at any rate of a part of the law, used to be given in the English text-books of thirty years ago—in fact we believe it still survives in Todhunter's *Analytical Statics*. We have never been able to discover anything about him, and as M. Duhem has not unearthed him in his investigations, we may safely conclude that he must have written after the first quarter of the 18th century. He may of course be quite a modern, but we have never yet met any one who can tell us anything about him. Perhaps a search through the earliest volumes of the *Nouvelles Annales* would reveal his identity. He does not so far (1799) occur in Cantor, Vol IV. Finally, as a token of the interest excited by the researches of M. Duhem we may add that the proofs were revised by Moritz Cantor.

Annuaire pour l'An 1909, publié par le Bureau des Longitudes. Avec Notices Scientifiques. Pp. 710, 116, 57, 11, 48. 1 fr. 50 c. 1908. (Gauthier-Villars.)

For 113 years the *Annuaire* has made its appearance, and it is interesting to conjecture what would be the feelings of its founders could they see the dimensions to which is has expanded. For it will be remembered that to possess the complete set of tables and lists dealing with every branch of science into which numerical data enter one must have the volumes of two consecutive years. This year it is the turn of the astronomer, the chemist, the physicist, and the engineer, and they will find exhaustive data suited to their respective needs. There are the usual essays on matters of immediate interest appended. M. Bigourdan writes on the Fixed Stars, and M. Lallemand on Movements and Deformations of the Terrestrial Crust. M. Radau's funeral oration over the late M. Janssen concludes the volume. The index runs to nearly fifty pages.

Histoire des Mathématiques. Par M. W. W. Rouse Ball. Translated into French by Lieut. L. Freund. Vol. I. vii, 422; Vol. II. pp. 270. 20 frcs. 1908. (Hermann, Paris.)

Mr. W. W. Rouse Ball's admirable *History* is now translated into French and Italian, and he is to be congratulated on this substantial recognition of the undoubted merits of that book. We could, however, have wished that the two large volumes published by Messrs. Hermann had more completely represented work done by Mr. Rouse Ball himself in the field in which he has made for himself a name. It is not that the additions which bring his volume of 520 English pages to

nearly 700 tall pages in the French edition are not full of interesting matter. Lieutenant Freund of the French Navy is responsible for those in the first volume, which take us up to Descartes and Huygens. Those in the second are by M. Montessus, and may be readily recognised by the asterisks in the text. The essays added to the first volume are as follows:—On Vieta as a geometer, a passage extracted from Chasles' *Histoire de la Géométrie*; an original analysis of the works of Napier relating to logarithms (28 pp.) by M. Biot; 5 pages on Kepler, taken from M. Bertrand's *Fondateurs de l'Astronomie*; The work of Galileo and Huygens, translated from Mach; and the preface of Duhem's *Origines de la statique*. Of these, the most interesting is the monograph by M. Biot. As it opens "Ayant été chargé il y a quelques mois. . . ." one might be led to suppose it is the work of recent investigation. But, at a guess, it is abstracted from Biot's *Mémoires Scientifiques*, consisting of his articles to the *Journal des Savants*. If this be so it is the only one of these notes that will not be found in the average mathematical library of any pretensions, the four large volumes being extremely rare and worth about £4 10s. The dates of the articles vary from 1830-1906. Nearly thirty pages on Vol. II. are given to M. Darboux's fine St. Louis lecture on *The Development of Geometrical Methods*. This is already within reach of English readers, being translated in the *Gazette* at the time, and afterwards in *Science*. We would have preferred some less accessible matter, such as, for example, Cantor's second Congress lecture on the *Historiography of Mathematics*, Hilbert's *Future Problems of Mathematics*, Poincaré's *Intuition and Logic in Mathematics*, or the like. The four pages of *errata* by no means exhaust those we have noticed, especially in English words, from Trinity "Collyge" to Sir "Olivier" Lodge. But we must not grumble. There is a good deal of additional matter in Vol. II., bringing the later portion of the history up to date. To those who are interested in the study of the history of mathematics we may cordially recommend the notices of this edition which have been appearing for some time past in the *Revue des Questions Scientifiques*, a Belgian journal of great merit. They are from the accomplished pen of the Père B. Lefebvre, S.J., and are marked by great erudition. We would suggest to M. Hermann that they should be reprinted as an addition to Vol. II. in the second edition. Though, as we say, they have been going on for some time, they so far have only reached the period of the Arabs.

An Elementary Treatise on the Differential Calculus founded on the Method of Rates. By W. W. Johnson. Pp. 191. 1908. (Wiley & Sons.)

Professor Johnson's *Treatise* has so recently been reviewed in these columns that there is no need to do more than call attention to the publication of an abridgment of the larger work, the main differences being in the earlier portions of the subject. The principal formulae of differentiation are established by more direct methods than the functional method in the *Treatise*. There are plenty of examples and yet not too many, which is an advantage to the private student who has no one to tell him that the existence of a set of fifty or sixty examples at the end of a chapter does not at all imply that they must be worked through before passing on to the next.

Lehrbuch der Differential- und Integralrechnung. By J. A. Serret. Third edition, revised and enlarged by G. Scheffers. Vol. I. Pp. xvi, 624. (Teubner.)

Serret's *Cours de Calcul Différentiel et Intégral* still seems to hold its own in the land of its birth and in Germany. It was published some thirty-five years ago, and Harnack's translation into German was issued in 1884. A second edition was published in 1904 by Bohlmann and Zermelo, and it concluded with about *four and a half pages of errata*. The last French edition came out in 1900 (it is time for another), and now we have a third German edition prepared by G. Scheffers. The book is transformed. One lamentable feature in its predecessors was the inferiority of the figures, and, what is unusual, the inaccuracy of many of them. These have disappeared; all the figures are new, and their number is increased. Typographical devices make the book much easier to read and to find one's way about in. Many of the chapters are entirely re-written and all are brought up to date. For instance, the sections on number are remodelled in accordance with the modern doctrine. More space is devoted to maxima and minima, and there are a couple of sections dealing with the functional determinant, while the inde-

pendence of functions and equations receives attention. There is a good ten-page index, and we can cordially congratulate the editor on the result of his labours. Serret's work is at last fit to take its place among trustworthy text-books on the subject. We have not yet received the second volume, on the Integral, and the third volume, which is to deal with Differential Equations, is not yet out. We may *en passant* venture to remark that the student who wishes to acquire German for his own or for examination purposes has now plenty of opportunities of doing so. He can try this volume with the new French edition when it appears. For more advanced work he now has translations of Forsyth's *Differential Equations* and of Osgood's *Theory of Functions*. For Mechanics he may tackle Mach with the aid of the translation published by the Open Court Co., and so on. It is astonishing how rapidly anyone with the slightest aptitude for linguistics can in this way pick up a running acquaintance with what we may call "mathematical German."

A Sequel to Elementary Geometry, with numerous Examples. By J. W. Russell. Pp. viii, 201. **Solutions to the Examples in A Sequel to Elementary Geometry.** (By the same hand.) Pp. 112. 1908. (Clarendon Press.)

Mr. Russell's volume, by its very title, challenges a reminiscent comparison with the late Dr. Casey's well-known *Sequel to Euclid*. School teachers will find it is very much better adapted for their purpose. Casey's volume contained a considerable amount of extraneous material, interesting enough in itself to a select few, but scarcely on the lines to be commended for youthful pursuit. To what one may call the leisured student it had its attractions, and Mr. Russell has, we think, done wisely in sweeping overboard almost the whole of what used to be called the "Recent Geometry of the Triangle." Indeed, the sections on this subject are confined to what is necessary to know w.r (to use his very acceptable abbreviation), isogonal lines and points, symmedians, antiparallels, and Brocard points. Let us say at once that in selection and arrangement of material the book bears the impress of the experienced teacher, and will be gladly welcomed by those who, while they want the irreducible minimum, yet want the phrase generously interpreted. The few analytical notes in the Appendix will be found useful. There is a plentiful supply of riders scattered throughout the different sections, and the book concludes with two carefully selected sets of "easy" and "harder" examples, in addition of course to those to be found at the end of each chapter. The value of the book is enhanced to many by the addition of a Key to all the Examples. Many of the solutions are models of the elegance we have a right to expect in such an acknowledged craftsman in the sphere of geometry as Mr. Russell has long shown himself to be.

The Physics of Earthquake Phenomena. By C. G. Knott. Pp. xii, 281. (Clarendon Press, Oxford.)

Dr. Knott's volume will be of interest both to the geologist and to the physicist, but portions of it will be as *caviare* to the general, unless they also happen to have some mathematics. The chapter on Seismometry contains an excellent discussion of the dynamics of the horizontal pendulum. We are not surprised that so shrewd a critic as Dr. Knott views with suspicion the use of harmonic analysis in the search for periodicities in a given body of statistics. His argument is as follows: Take a group of statistics of earthquakes for the months of the year. Measure on a horizontal line twelve regular intervals, and at the centre of each interval draw to scale a vertical line representing the corresponding frequency. This gives a rough graph showing the march of frequency with time during the year. With the aid of Fourier's theorem, we analyse this function into simple harmonic components, the periods of which are 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. The period of the first is one year, of the second six months, of the third four, and so on. This is "a purely mathematical operation enabling us *simply* to build up the original function by putting together the components again, or, if it has any physical significance at all, it is based implicitly on two very doubtful assumptions: (1) that the acting causes themselves are simple harmonic in their variation, and (2) that this simple harmonic quality is *dynamically reproducible* in the system acted upon. The whole theory of forced vibrations . . . shows

how unwarranted this latter assumption may be. . . . I am convinced that sufficient accuracy is attained by use of the purely arithmetical overlapping summations over suitable intervals." Dr. Knott's admirable volume fills a gap, for we badly wanted a text-book from competent hands dealing with these physical investigations, and treating the subject "as belonging to the . . . domain of natural philosophy, both experimental and mathematical." This service Dr. Knott has adequately performed.

The Collected Papers of J. J. Sylvester. Vol. II. (1854-1873). Edited by Dr. H. F. Baker. Vol. II. Pp. xvi, 731. 1908. (Cam. Univ. Press.)

J. J. Sylvester died on Mar. 15th, 1897, two years after Cayley and less than a month after Weierstrass. "I could wish," said Prof. Elliott in November of that year, "that the worthiest of memorials could be raised to him by the collected publication of all his contributions to the existing sum of mathematical thought, with their idiosyncrasies, their marks of his unique personality still upon them as they came from his pen, or, at any rate, as they left the perplexed compositor." Part of this wish is well on to the completion of fulfilment. Major MacMahon has told us, in the graceful tribute written for *Nature* (Mar. 25th, 1897), that Sylvester's handwriting was bad and a trouble to the printers. He tells us how no sooner was the manuscript in the editor's hands than alterations, corrections, ameliorations, and generalisations would suggest themselves to his mind, and every post would carry further directions to the editors and printers. And, again, Prof. Elliott alluded to the blemishes which tried printers and editors to the limits of their endurance, the impatient fitfulness, the restless haste of production, the supplements and restatements, the digressions from which there was no return—all of which he mentions as signs of a vital force to which it is hard to find a parallel, a vital force which vivified the energies of all who came under its direct influence. It has not, however, been thought fit that the whole of Prof. Elliott's wish should be gratified. The work of editing has been entrusted to the competent hands of Mr. Baker. Misprints no doubt will be found by critical eyes, but we confess to have noticed but one—in Vol. I., p. 382 —where a "strange negligence" becomes a "strangle negligence." The period covered by the second volume is that in which Sylvester established his reputation "as the foremost of living mathematicians." In Newton's *Arithmetica Universalis* we find a rule for the discovery of imaginary roots of equations and for assigning an inferior limit to their number. Of the process by which he arrived at the rule we are given no hint. It may have been a case of the intuition of genius, as with Fermat's last theorem, for in each instance there is no doubt that intuition did not fail the respective authors. Sylvester's great *Memoir* of 1864 established the validity of Newton's rule for finding an inferior limit for imaginary roots of equations as far as the fifth degree inclusive. The next year he gave an elementary proof and generalisation of the "marvellous and mysterious rule," and asserted that all previous efforts to the elucidation of the problem has been confined exclusively to an imperfect form of Newton's theorem. This brought him into conflict with Dr. J. R. Young, who claimed to have established the rule some twenty years before. Nothing ever aroused Sylvester's ire so much as a reflection on his claims to priority of mathematical discovery. In a page and a half he demolishes the pretensions of the unhappy claimant, concluding his *exposé* as follows: "This is the sum and substance, in a simplified form, of Dr. Young's so-called proof. It is 'such stuff as dreams are made of,' and, culminating as it does in a palpable *petitio principii*, does not need a detailed refutation at the hands of the author of this lecture. It is not by such vague rhetorical processes, but by quite a different kind of mental toil, that the truths of science are to be won, or a way opened to the inner recesses of the reason." The Presidential Address to the B.A. at Exeter is also included in this volume, and should prove of considerable interest to readers of the *Gazette* at the present moment, containing as it does a vigorous defence of mathematics as a science of observation and experiment. Huxley had recently strayed from the beaten track in which he was second to none of his contemporaries, and, in what he called "An After-Dinner Speech," he had permitted himself to observe that "mathematical reasoning is almost purely deductive," and that, as Sylvester puts it, to make out a mathematical proposition and to construe or parse a sentence, are equivalent or

identical mental operations. Sylvester could not resist the gibe that his distinguished friend would have couched his statement in more guarded language had his speech been made *before* dinner instead of *after*. *Apropos* of the value of the opinion of experts in one branch of knowledge when they come to deal with other matters which do not as a rule come under their survey, it may be well to note the aptness of quotation characteristic of Sylvester. From Goethe he took the lines

Verständige Leute kannst du irren sehn
In Sachen, nämlich, die sie nicht verstehen.

Understanding people you may see erring—in those things, to wit, which they do not understand. And again, from Cuvier's *éloge* of Daubenton: "Les savants jugent toujours comme le vulgaire les ouvrages qui ne sont pas de leur genre." And as Major MacMahon remarks, Huxley, "like many other remarkable men in other branches of science, had no conception of the real nature of the life-work of mathematicians of the high order to which Sylvester belonged." Under the circumstances the following passage may be deemed singularly happy. "The New Algebra originated in the accidental observation by Eisenstein, some score or more years ago, of a single invariant (the Quadrinvariant of a Binary Quartic) which he met with in the course of certain researches, just as accidentally and unexpectedly as M. du Chaillu might meet a Gorilla in the country of the Fantees, or any one of us a White Polar Bear escaped from the Zoological Gardens. Fortunately he pounced down upon his prey, and preserved it for the contemplation and study of future mathematicians. It occupies only a page in his collected posthumous works. This single result of observation (as well entitled to be so called as the discovery of Globigerinae in Chalk or of the confoco-ellipsoidal structure of the shells of the Foraminifera), which remained unproductive in the hands of its distinguished author, has served to set in motion a train of thought, and to propagate an impulse which has led to a complete revolution in the whole aspect of modern analysis, and whose consequences will continue to be felt until Mathematics are forgotten and British Associations meet no more." In nearly all of the papers in the two volumes one is struck with the illimitable enthusiasm which must have made Sylvester such a fascinating personality to those with whom he was brought in contact. "At times," said Prof. Forsyth, "his mind became so possessed with his sense of the high Mission of Mathematics, that his utterances had the lofty note of the prophet and the seer." We know of no man of science who so naturally conveys to the reader the impression that he was born into the world *natura ipsa valere, et mentis viribus excitari, et quasi quodam divino spiritu afflari.*

Leçons d'Algèbre et d'Analyse. By J. Tannery. Vol. I., pp. viii, 423; Vol. II., pp. 636. (Gauthier-Villars, Paris.)

Books that are written with the view of covering a syllabus in most cases suffer considerable disadvantage. Prof. Tannery's *Leçons* are designed to supply the needs of the pupils of the *Classes de Mathématiques Spéciales*. In other words they are meant for those who require certain portions of Algebra and Analysis for the purpose of applying them in the immediate future. He therefore feels it incumbent upon him to write in apologetic terms of the lack of rigour that may at times be perceptible in his treatment of parts of his subject. When he has omitted certain arguments which are indispensable to the establishment of a theorem in logical rigour, he warns the reader of the fact, and points out the gap that is left and the difficulties which are passed over. It is characteristic of the man that he adds: "I have a horror of teaching what is not sincere; respect for truth is the first if not the only moral lesson that is to be learned in the study of the sciences. No doubt there are proofs which are not rigorous and that are excellent, because they leave an image on the mind which is not effaced. Both in practical life and in theoretical investigation it is essential to distinguish between what we know we understand and that of which one feels one has every reason to be persuaded; it is well to distinguish what one possesses in its entirety from what one may use under certain conditions." There is some point also in his justification of himself for laying no stress on the historical development of his subject. He thinks it would be a pity that his classes, consisting of students who are not going to be mathematicians, should only know of Descartes that he found a certain rule of signs, of Newton that he invented a formula of approxima-

tion and had something to do with the binomial theorem, and that after covering the ground laid out in these volumes they should be tempted to think that Rolle was a mathematician of eminence equal to that of Descartes or Newton. And it is also interesting to see what he considers the ideal teaching. If the student who has assimilated it completely is astonished at the natural manner in which he retains fundamental principles, the theories deduced from them, the methods resulting from them, then that kind of teaching is the ideal, for the principles are so clear, the deductions so natural, and the methods so easy, that they can at any moment be recalled without effort. As far as style and lucidity of exposition are concerned, Prof. Tannery has certainly inherited the distinctive gift of his race.

But to come to close quarters. The first chapter unfolds the modern doctrine of irrational numbers. Definitions and operations are rigorously treated and made clear by appropriate devices. The practical import of what has been introduced to the student is then enforced by the extension of the ideas propounded to arcs and areas and by a set of carefully selected exercises. The next two chapters are devoted to polynomials with real variables, and in chapter VII., after the discussion of imaginary numbers, we have an extension to polynomials of which coefficients and variables may be imaginary. Graphical illustrations facilitate a mastery of the sections on rational fractions, and they terminate with an excellent study of the homographic function $y=\dfrac{ax+b}{a'x+b'}$.

In the chapter on H.C.F. and L.C.M. the author works out an example in the ordinary way without detached coefficients. It fills three-quarters of a page. The writer finds the following method satisfactory with beginners (v. *Chrystal*, Introduction, p. 202).

$$U_1 \equiv 2x^4-8x^3+9x^2-4x+4\,; \qquad U_2 \equiv x^4-3x^3+2x^2-4x+8.^*$$

$$V_1 = 2U_2-U_1 \equiv 2x^3+5x^2-4x+12\,; \quad V_2=\frac{1}{x}(2U_1-U_2)\equiv 3x^3-13x^2+16x-4\,;$$

$$\therefore\; V_1+3V_2 \equiv 11x(x^2-4x+4)\equiv 11x(x-2)^2.$$

Neither 11 nor x is a factor of U_1 and U_2. Hence if there is a common factor other than unity $x-2$ or $(x-2)^2$ is that factor.

$$U_1=(x-2)(2x^3-4x^2+x-2) \quad \text{(Use the remainder theorem.)}$$
$$=(x-2)^2(2x^2+1),$$
so,
$$U_2=(x-2)^2(x^2+x+2).$$

The two methods are of course identical, but to young students the *rationale* of the repeated divisions is obscured in the process of working out, whereas the underlying principle is never lost sight of while repeatedly applying "whatever is a factor of A and B is a factor of $mA\pm nB$." There is, too, a further advantage in that (i) the remaining factors are exhibited, and (ii) the L.C.M. can be read off.

The second example worked out is even simpler still, though beginners will not feel quite as happy in dealing with literal coefficients. It takes over half-a-page, and the pages are about 7 in. by $5\frac{1}{2}$ in.

$U_1\equiv x^3+px+q\,;\ U_2\equiv 3x^2+p\,;\ (p\neq 0)$. Find the condition that U_1 is prime to U_2.

$3U_1-xU_2\equiv 2px+3q$. This is a factor of U_2 if $3\left(-\dfrac{3q}{2p}\right)^2+3q=0$,

i.e. if $27q^2+4p^2=0$ or if $q\neq 0$. Similarly for U_1.

Hence U_1 is prime to U_2 if $27q^2+4p^2\neq 0$.

The chapter on linear equations is of more than ordinary interest from the attention paid to the superior and inferior limits of the errors resulting from approximations when the cofficients are unwieldy, as must occur in practice. The slide rule or tables may be used. We give the typical equations worked out by the author, the coefficients of which were selected at random:

$$3275x-8243y-7425=0=6287x+2724y-6232.$$

His method can of course be applied to the numerical solution of n equations with n unknowns.

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* There is a misprint here of $-2x^2$ for $+2x^2$.

The second volume opens with a chapter on Series, in which the syllabus has evidently restricted the author to the most elementary treatment, Cauchy's convergence condition being relegated to a footnote. Still, what there is in the chapter is sound and clearly expressed. Summing to a given degree of approximation where an approximate value for each term may be taken is treated at length with illustrative examples. Functions of a real variable (plentifully illustrated by graphs) and differential coefficients are followed by a chapter on series of functions in which numerical calculations again receive due attention. Fifty pages are then devoted to the variations of given functions, separation of and approximations to the roots of an equation. In the chapter on algebraical equations the author emphasises the distinction to be drawn between symmetric functions of n variables and those of the n roots of an algebraical equation, *e.g.* $x_1^2-x_2x_3$ is a symmetrical function of the roots of the equation $x^3+px+q=0$. Here also an interesting section is devoted to the theorem: the calculation of a rational symmetrical function $F(x_1, x_2 \ldots x_n)$ of the roots of $A_0x^n+A_1x^{n-1}+\ldots=0$ reduces to that of two polynomials symmetrical with respect to the variables $x_1, x_2 \ldots x_n$. This chapter makes ample use of Waring's method. The chapter on the Integral Calculus consists of about 150 pages, of which 10 are devoted to approximate evaluations of a definite integral and 20 to differential equations.

The teacher will derive much benefit from, in particular, the small print paragraphs scattered throughout the book. His greatest difficulty is always to put himself in the place of one who is at a lower stage of intellectual development, to anticipate his doubts, to be at hand where he may fall, to guide his rising curiosity into fruitful ways, to see that he gets the habit of cautious thinking, of not taking things for granted without knowing he is doing so—in all these and in many other ways, Prof. Tannery's thoughtful volumes will, we feel sure, instruct and stimulate.

NEW EDITIONS.

Mr. Alfred Lodge is to be congratulated in that the demand for his *Differential Calculus for Beginners* has already justified the issue of a third edition. The book sadly needed a set of Miscellaneous Examples for revision purposes, and this want has now been supplied. Mr. Baker's *Elementary Dynamics* has also found its *clientèle*, and has likewise reached a third edition. It is now, as far as we can see, entirely free from the minor blemishes to be found in early editions. For the information of the private student and others, we may add that a full *Key* to the examples has been published at the prohibitive price (to the small boy) of 10s. 6d. net. There is no need to say more of the merits of books the peculiar qualities of which are so widely recognised. Both volumes are published by Messrs. Bell.

QUERIES.

(63) What is the mathematical theory of the "Hatchet Planimeter" described in Castle's *Practical Mathematics for Beginners*, Macmillan, 1901, p. 237? "Goodman's Patent Planimeter" and "Averaging Instrument," advertised in *Nature* for July 9, 1908, appear from the figure to be identical with this. G. H. B.

(64) Has anyone constructed a "spectrum" of prime numbers by taking a scale, say of millimetres, to represent different odd numbers, and indicating the primes by dark or, if preferred, white lines resembling the lines on a spectrum? PRYMO.

(65) Is the differential equation of the family of curves $dx/ds=ay+by^{-\frac{1}{2}}$ integrable in terms of elliptic or other known functions? PHUGOID.

(66) The following, from Halley's *Catalogus stellarum australium* (1679), may supply an interesting problem for an unoccupied half-hour.

Requiritur vero solutio Problematis cujusdam Geometrici quamquidem ipse irrito labore attentavi: Geometras itaque insignes impensius oro, Astronomiam eorum opem efflagitantem adjuvare velint, Problemaque insequens solutum dent.

Datis tribus Circulis quorum centra sunt in Linea recta, oportet in eorum Circumferentiis, trianguli, triangulo dato similis, angulos collocare, ita ut trianguli latera sub datis Angulis ad lineam per centra Circulorum inclinentur.

Perhaps one of our readers may be able to give some historical information as to solutions. E. M. L.

ANSWERS TO QUERIES.

[**iv. 54, p. 295.**] In the given curve let n=degree, m=class, D=deficiency, δ, κ, τ, ι=number of nodes, cusps, bitangents, inflexions. The degree of the pedal (the inverse of the polar reciprocal)$=2m$; or $=2(m-1)$ if the pole is a focus of the given curve.

Now $2(m-1)-n=\iota-2D$. Suppose the curve to be derived by gradual change of the coefficients from one in which $D=0$. Each time a double-point disappears $2D$ is increased by 2, while ι is increased by 6 (at least); therefore $\iota-2D>0$, unless $\iota=D=0$. Hence the degree of the pedal$=n$ only when n is even and the pole is a focus of the given curve, while $m=\frac{1}{2}(n+2)$, $\delta=\frac{1}{8}(n-2)(n-4)$, $\kappa=\frac{3}{2}(n-2)$, $\tau=\frac{1}{8}n(n-2)$, $\iota=D=0$.

Exceptional cases are not here considered, *e.g.* when the pole is the intersection of two bitangents each of which passes through a circular point at infinity. HAROLD HILTON.

[**iv. 8, p. 95.**] Now I, even I would celebrate
In rhymes inapt, the great
Immortal Syracusan, rivalled nevermore,
Who in his wondrous lore,
Passed on before,
Left men his guidance how to circles mensurate.

A. C. ORR in the *Literary Digest.*

Now, O hero, great advancing in method
Which you would proclaim wonderful, worketh universal;
Yet in our memories your labour is rooted;
Unto the end should'st thou be amongst immortals.

R. D. CARMICHAEL.

BOOKS, ETC., RECEIVED.

Elemente der Matematik (Jules Tannery). Translated by P. KLAESS. With an introduction by F. KLEIN, and an historical Appendix by PAUL TANNERY. Pp. xii, 339. 7 m. 1909. (Teubner.)

Die Elemente der Geometrie. By H. THIEME. A. *Planimetrie.* B. *Stereometrie.* Pp. xii, 394. 9 m. 1909. (Teubner.)

Anfangsgründe der Maxwellschen Theorie, verknüpft mit der Elektronentheorie. By F. RICHARZ. Pp. ix, 245. 7 m. 1909. (Teubner.)

Zurückführung der sphärischen Trigonometrie auf die Geometrie des ebenen Kreisvierecks. By E. ECKHARDT. Pp. vi, 155. 4·40 m. 1909. (Teubner.)

Die Lehre von den Geometrischen Verwandtschaften. III. *Die eindeutigen linearen Verwandtschaften zwischen Gebilden dritter stufe.* Pp. viii, 574. 1909. (Teubner.)

Formulario Mathematico. By G. PEANO. Editio V. Tome V. of the complete edition. 20 lire. 1908. (Bocca, Turin.) Preface and Bibliography. XXXVI.

VI. Differential Calculus. Pp. 275-338. VII. Integral Calculus. Pp. 339-388. VIII. Applications to Geometry and Complement. (A reproduction of Dr. G. Pagliero's Theory of Curves, 1905-6.) Pp. 389-459.

Cassell's Elementary Geometry. By W. A. KNIGHT. Pp. vii, 253. 1909. (Cassells.)

An Elementary Treatment of the Theory of Spinning Tops and Gyroscopic Motion. By H. CRABTREE. Pp. xii, 140. 5s. 6d. net. 1909. (Longmans, Green.)

Nouvelles Annales de Mathématiques. Edited by MM. LAISANT, BOURLET and BRICARD. May, 1909. 17 frcs. per ann. (Gauthier-Villars.)

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